Consensus dynamics for hypergraphs: Hodge Laplacian and group reinforcement model

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Multi-body interactions can reveal higher-order dynamical effects that are not captured by traditional two-body network models. In this work, we derive and analyse two models for generalizing consensus dynamics on hypergraphs, where nodes interact in groups rather than in pairs. The first framework uses the Hodge Laplacian of a simplicial complex to define linear dynamics on it. We find its equilibrium points and relate them to its topology through the simplicial homology. The second generalization, incorporates reinforcing group effects, which defines non-linear dynamics in an hypergraph. We show that unlike consensus dynamics, the mean field may shift with time. However, some properties are preserved, as they have the same equilibrium points and every orbit tends to one of them. With numerical simulations we see that, despite the undirected nature of this structure, the high-order interactions can create directional dynamics.

Hypergraph| Consensus dynamics | Hodge Laplacian | Simplicial complex

Networks provide a powerful framework for modelling interacting systems, as they capture the essence of connectivity. Its strength comes from its minimalism and generality, only dealing with two node interactions. However, in recent years, the need for models where higher order interactions are allowed has been apparent (1). Areas where this structure is needed include collaboration of authors (2) and neural activity (3). The main mathematical objects used to study these interactions have been hypergraphs (4, 5) and the more restrictive, simplicial complexes (6, 7).

One of the main areas of study within networks is their dynamics, where time-varying states are assigned to the nodes and evolve according to interaction rules defined between neighbouring. Sufficiently simple for theoretical investigations, the resulting dynamics may exhibit complex global behaviour, making them suitable to model various real-world systems (8). Despite the importance of this subfield, the study of how multi-body interactions in an hypergraph affect the spreading dynamics is still nascent, see (6, 9) for discrete dynamics and (10, 11) for continuous ones.

The goal of this paper is to generalize the simplest dynamics defined on a graph, the consensus dynamics, to the context of hypergraphs. Our first proposal uses the Hodge Laplacian, which is the equivalent of the Laplacian matrix of a graph, when dealing with a simplicial complex. This has already been done in the context of discrete dynamics and random walks in (6), but to the best of our knowladge it has not been used to define continuous dynamics in a simplicial complex. Our main result for this model is the location of the equilibrium points and their relation with the homology of the simplicial complex.

Our second generalization builds on (11), where dynamics in hypergraphs with exclusively 3 node interactions is proposed. This model is theoretically less appealing than the first one as it is not as directly related to consensus dynamics. However, it is much easier to implement in practice since it is nodecentric, like most real phenomenons. That is, state variables in this model are exclusively in nodes, whereas in the first model we have state variables for simplexes of all orders.

We first expand the work on (11) to be able to deal with a generic hypergraph. The dynamics in this framework are much harder to understand theoretically, as they are non-linear. For instance, unlike consensus dynamics, the mean field may shift with time. Despite this, we are able to show that its equilibrium points coincide with the ones of consensus dynamics, and that any orbit converges to an equilibrium point. This result was not know even for the case with exclusively 3-way interactions. In the final section we perform some numerical simulations for this model and we show how, even with the undirectional structure of the hypergraph, the higher-order interactions allow us to create directional like dynamics. This had already been shown in (11) with 3-way interaction, but when using higher-order ones, the set up for directional dynamics can be simplified.

Basic definitions

An hypergraph $\mathcal{H}$ is given by a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices or nodes, and $\mathcal{E}$ is a subset of the powerset $P(\mathcal{V})$, and its elements are called edges. In this paper we will always assume that the set of nodes is finite. Given an edge e and a node n of an hypergraph we will abusively denote $e \in \mathcal{H}$ and $n \in \mathcal{H}$. We say that an edge is a k-edge if it has cardinal k. A graph is simply an hypergraph formed exclusively by 2-edges. We say that two nodes of an hypergraph are adjacent if there is an edge which contains both of them. With this notion we can define connectivity and connected components in an hypergraph as we do in graphs. A more extensive introduction to this mathematical construction is given in (5).

Significance Statement

Networks dynamics provides useful models for many natural and man-made phenomena, such epidemic spreading, elections and power grids. The main example of these is consensus dynamics, which given an initial configuration, continuously changes it until reaching an equality between the nodes. Recently, it has been noted that network models, which are based in two-body interactions, are not suited to model some phenomenon where higher order interactions appear. Since then there has been an increasing attention to hypergraphs, a generalization of networks where multibody interactions are allowed. We propose two dynamics on hypergraphs, that generalize the consensus dynamics on a network.

A special subclass of hypergraphs are called simplicial complexes, which are widely used in algebraic topology. These have the additional condition that any subset of an edge is also an edge. In this context we refer to edges as simplexes (although many authors use the term face), and k-simplexes correspond to $(k+1)$-edges. Although graphs are not formally simplicial complexes, as they lack 0-simplexes, we can always add these simplexes to the set of edges and form a simplicial complex which encodes the same information as the original graph.

Simplicial complexes have been hugely influential in mathematics due to the existence of a boundary map on them, which enables the computation of their homology. We proceed to do a brief overview of this construction, a more detailed explanation is given in (12). Let $\mathcal{S}$ be a simplicial complex. We fix a ordering of its vertices (nodes) which induces an orientation on its simplexes. Then, we denote by C_k the $\mathbb{R}$-vector space generated by the oriented k-simplexes of $\mathcal{S}$. Then, there is a boundary map $d_k : C_k \to C_{k-1}$ (see (12) for definition) such that $(C_\bullet, d_\bullet)$ is a chain complex. If we also consider the maps $d_\bullet^T$ we get,

$$0 \longleftrightarrow C_0 \underset{d_1^T}{\overset{d_1}{\longleftrightarrow}} C_1 \underset{d_2^T}{\overset{d_2}{\longleftrightarrow}} C_2 \underset{d_3^T}{\overset{d_3}{\longleftrightarrow}} \cdots \qquad [1]$$

Note that $d_\bullet^T$ can be thought of as the coboundary operator. As $d_\bullet$ is a boundry operator it satisfies $d_k \circ d_{k+1} = 0$ and hence $\text{im } d_{k+1} \subset \ker d_k$. Thus, we can define the kth homology vector space as,

$$H_k(\mathcal{S}, \mathbb{R}) = \text{im } d_{k+1} / \ker d_k.$$

It is well known that these vector spaces encode many topological information about $\mathcal{S}$, see (12).

Hodge Laplacian Dynamics

Consensus dynamics on a graph G with N nodes is defined over the state space $\mathbf{x} \in \mathbb{R}^N$, where each node has the scalar state x_i, by the ODE,

$$\dot{\mathbf{x}} = -L\mathbf{x},$$

where L is the Laplacian of the graph G. Hence, if we can generalize the matrix L for simplicial complexes we will have an obvious generalization of concensus dynamics for them. To do so we will use the boundary maps introduced in the previous section.

Let $\mathcal{S}$ be an oriented simplicial complex with N nodes. Using the maps in Eq. (1) we can define the Hodge k-Laplacian of $\mathcal{S}$ as,

$$\mathcal{L}_k = d_k^T d_k + d_{k+1} d_{k+1}^T.$$

This is a generalization of the standard Laplacian of a graph, as it is well known that $L = d_1 d_1^T$ and that $d_0 = 0$, so we get $\mathcal{L}_0 = L$.

Now we define the Hodge Laplacian dynamics in $\oplus_{k=0}^N C_k$ as the set of decoupled ODE,

$$\dot{\mathbf{x}}_k = -\mathcal{L}_k \mathbf{x}_k,$$

where $\mathbf{x}_k \in C_k$. Note that the dynamics over $\mathbf{x}_0$ are exactly the consensus dynamics over the graph that the 0-simplexes and 1-simplexes create.

The Hodge k-Laplacian is a symmetric matrix positively semi-defined. Indeed,

$$\mathcal{L}_k^T = (d_k^T d_k)^T + (d_{k+1} d_{k+1}^T)^T = d_k^T d_k + d_{k+1} d_{k+1}^T = \mathcal{L}_k,$$

and for all $\mathbf{x} \in \mathbb{R}^N$,

$$\mathbf{x}^T \mathcal{L}_k \mathbf{x} = \mathbf{x}^T d_k^T d_k \mathbf{x} + \mathbf{x}^T d_{k+1} d_{k+1}^T \mathbf{x} = ||d_k \mathbf{x}||^2 + ||d_{k+1}^T \mathbf{x}||^2 > 0.$$

Thus, its dynamics are similar to the consensus ones. Any initial condition $\mathbf{y}_k \in C_k$, can be expressed as the sum of eigenvectors for the different eigenvalues of $\mathcal{L}_k$. Then, as $-\mathcal{L}_k$ is negative semi-definited, all the components corresponding to non 0 eigenvalues will tend to 0 when t go to infinity. Using the fact that $\mathcal{L}_k$ is symmetric, and hence it has orthogonal eigenvectors for different eigenvalues, we get that, for all solution $\mathbf{x}_k(t)$,

$$\lim_{t \to \infty} \mathbf{x}_k(t) = p_k(x_k(0)), \qquad [2]$$

where $p_k : C_k \to \ker \mathcal{L}_k$ is the orthogonal projection to the null space. For $k = 0$, we have that $\ker \mathcal{L}_0$ is generated by the indicator vectors of the connected components. If $\mathcal{S}$ is connected we get $\ker \mathcal{L}_0 = \langle (1, \ldots, 1)^T \rangle$, and Eq. (2) reduces to the well known fact that in consensus dynamics all components of a solution tend to the average of the initial state components. For general k, as shown in (13) we have,

$$\ker \mathcal{L}_k = \ker d_k \cap (\text{im } d_{k+1})^\perp \cong H_k(\mathcal{S}, \mathbb{R}),$$

so the null space has a strong topological interpretation. For instance, $\ker \mathcal{L}_1$ is the space generated by the cycles which are not the boundary of a combination of 2-simplexes.

Another interesting property of Hodge Laplacian Dynamics is that they are the square of the obvious dynamics that one may define from Eq. (1), in SI Section A we develop this further.

Drawbacks of the model. An obvious limitation of this model is that it can only be used with simplicial complexes, and some hypergraphs from empirical data do not have this additional structure.

However, the main obstacle when trying to use this model is that often in empirical data, state variables can only be measured in nodes. This poses challenges in both analysis and interpretability, as our model give states variable to all simplexes.

Group reinforcement model

In (11) it is studied a model to generalize consensus dynamics for hypergraph formed exclusively of 3 edges. Following the comments from the Discussion of the aforementioned paper we can extend this construction for a generic hypergraph as follows.

Given a hipergraph $\mathcal{H}$ with node states x_i for $i \in \{1, \ldots, N\}$ we define,

$$\mathcal{H}_i = \{A \setminus \{i\} : A \in \mathcal{H} \text{ and } i \in A\}.$$

Then we define the following ODE,

$$\dot{x}_i = \sum_{A \in \mathcal{H}_i} \sum_{j \in A} s\left(\left| x_j - \frac{\sum_{k \in A} x_k}{|A|} \right| \right) (x_j - x_i), \qquad [3]$$

where s is a arbitrary scalar non-negative function, which we will always choose monotonic. Note that if $\mathcal{H}$ is a graph and

$s(0) = 1$, we recover the consensus dynamics. We introduce the non-linear effect of s since we will show that if s is constant, then the dynamics above can be viewed as the consensus dynamics of a weighted graph in the nodes $\{1, \dots, N\}$. Hence, to observe a genuine effect of the high-order structures, which cannot be reduced to binary interactions, we need to introduce the non-linear factor s.

We can see $s(|x_j - |A|^{-1} \sum_{k\in A} x_k|)$ as modulating the intensity of the effect that the state x_j has on x_i. Note that if s is non-increasing, then the effect of x_j on x_i is reinforced if x_j is close to the average state in A (for instance if all states in A coincide) and hindered if its far from the average in A. This property is reminiscent of non-linear voter models (14) where the voter changes his opinion with a probability that depends non-linearly on the fraction of disagreeing neighbours.

Note that when A comes from an edge with 2 nodes, we have $|A| = 1$ and hence $s(|x_j - |A|^{-1} \sum_{k\in A} x_k|) = s(0)$. If s is non-increasing, this is the maximal value it can take, and hence we are giving more imporatance to 2-way interactions than to other types of interactions. To avoid this, we may replace s in Eq. (3), for $s_{|A|}$, where s_k are different functions that model each order of interactions. By choosing this functions appropriately we can make certain order of interactions to have more importance. Then, our model can exhibit effects such as the ones predicted by Sznajd model (15), which claims: An isolated person does not convince others; a group of people sharing the same opinions influences the neighbours much more easily. For simplicity, in what follows we will restrict ourselves to the case when $s_l = s$, but analogous study can be done for the general one.

Theoretical results. First we show that the ODE given by Eq. (3) can be viewed as the consensus dynamics on a weighted graph, where the weights depend on the position. Indeed, given $\mathbf{x} \in \mathbb{R}^N$ we define the matrix,

$$[W(\mathbf{x})]_{i,j} = \sum_{A:\, j\in A\in\mathcal{H}_i} s\left(\left|x_j - \frac{\sum_{k\in A} x_k}{|A|}\right|\right),$$

where we note that $W(\mathbf{x})$ is not symmetric and has null diagonal entries. Then, reordering the sums in Eq. (3) we get,

$$\dot{x}_i = \sum_{j\neq i} (x_j - x_i)\, [W(\mathbf{x})]_{i,j}\,.$$

Now if we denote by $L(\mathbf{x})$ the associate Laplacian of the weighted adjacency matrix $W(\mathbf{x})$ we can reduce Eq. (3) to,

$$\dot{\mathbf{x}} = -L(\mathbf{x})\mathbf{x}, \qquad [4]$$

If s is constant, then L is constant, and the equation above shows that we are dealing with consensus dynamics on a weighted graph (this is essentially the same argument done in (11) for 3-edges). Hence, in this case we are not dealing with proper hypergraph dynamics. When s is not constant, W changes from point to point, thus it can not be studied as consensus dynamics. However, we can still use Eq. (4) to find its equilibrium points.

Equilibrium points. Suppose for simplicity that $\mathcal{H}$ is connected and s is positive (similar results can be shown for the general case). Then, for all $\mathbf{x}$, $L(\mathbf{x})$ is the Laplacian matrix of a weighted strongly connected graph, and hence its null space is generated by $(1, \dots, 1)^T$ (see (16)). Thus, by Eq. (4), we have $\dot{\mathbf{x}} = 0$ if and only if $\mathbf{x} \in \langle (1, \dots, 1)^T \rangle$.

Convergence. We now proceed to show that if $\mathcal{H}$ is connected and s is positive, all orbits converge to an equilibrium point. Let $\mathbf{x}(t)$ be a solution of the ODE given by Eq. (3), then if $x_{\max}(t)$ is the maximal coordinate at t, all terms in Eq. (3) are non-positive, and hence $\dot{x}_{\max}(t) \leq 0$. The same argument yields $\dot{x}_{\min}(t) \geq 0$. Thus, we have a family of closed intervals parametrisied by $t \in \mathbb{R}_{\geq 0}$,

$$I_t = \left[\min_i x_i(t), \max_i x_i(t)\right],$$

such that $I_t \supset I_s$ if $t \leq s$. Then, $I = \cap_{t\in\mathbb{R}_{\geq 0}} I_t$, is not empty, closed and connected, thus $I = [a, b]$ for some $a, b \in \mathbb{R}$. We prove that $a = b$, which implies that $\mathbf{x}(t)$ converges to the equilibrium point. Assume $a < b$, then

$$\frac{d}{dt}(\min_i x_i(t)) > (b - a)\delta > 0,$$

where,

$$\delta = \max_{x\in I_0} s(|2x|),$$

which is bigger than 0 as s is positive. This is a contradiction as it implies that $\min_i x_i(t)$ goes to infinity as $t \to \infty$, but it also has to be in I.

Note that the argument above also shows that if all components are in an interval of tolerance, then the convergent point of the orbit will also be in this interval.

Mean field. An important property of consensus dynamics is that the average state $\overline{\mathbf{x}}$ is constant over orbits, i.e. $\sum_{i=1}^N x_i$ is a first integral of the system. This is no longer true in this context due to the non-linearity introduced by s, we have,

$$\dot{\overline{\mathbf{x}}} = \frac{1}{N} \sum_{\substack{B\in\mathcal{H} \\ |B|\geq 3}} \sum_{\substack{i,j\in B \\ i\neq j}} s\left(\left|x_j - \frac{\sum_{k\in B\setminus\{i\}} x_k}{|B| - 1|)}\right|\right)(x_j - x_i).$$

Note that the effects from 2-edges do not appear in the expression above, as all of them have the modularity factor $s(0)$ and hence they cancel each other as happens with consensus dynamics. Alternatively, we can use the expression of $\dot{x}_i$ in terms of the entries of $W(\mathbf{x})$ to get,

$$\dot{\overline{\mathbf{x}}} = \frac{1}{N} \sum_{i,j=1}^N (x_j - x_i)\, [W(\mathbf{x})]_{i,j}\,. \qquad [5]$$

Trivially, if $W(\mathbf{x})$ is symmetric, we get $\dot{\overline{\mathbf{x}}} = 0$. If we have certain symmetries in the initial conditions and in the topology, we will have that $W(\mathbf{x}(t))$ is symmetric for all times, and hence $\overline{\mathbf{x}}$ constant. We will see this happening in several numerical simulations.

To finish this section we mention that it is easy to check that the field defined by Eq. (3) has negative divergence at every point.

Numerical simulations. We choose $s(x) = e^{-\lambda x}$ as it is maybe the simplest, positive, decreasing function. Moreover, it is used in many nature and sociology models. We will always choose $\lambda = 1$ if not stated otherwise.

To start with, we do simulations on the fully connected hypergraph, i.e. $\mathcal{E} = P(\mathcal{V})$, with 8 nodes. Starting with a random initial condition (which we will always assume to be chosen uniformly in the range $[0, 1]$ for each coordinate) the

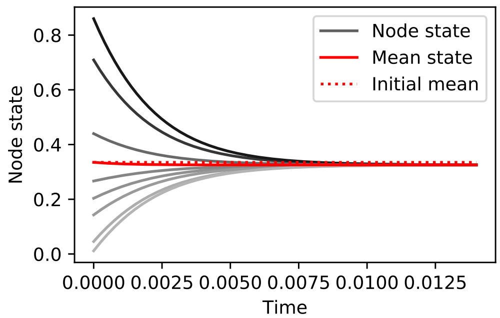

Fig. 1. Coordinates of an orbit in group reinforcing dynamics for a complete hypergraph of 8 nodes, with $\lambda = 1$. The initial condition is chosen at random and the grey's intensity increases with the value of the initial condition of the coordinate.

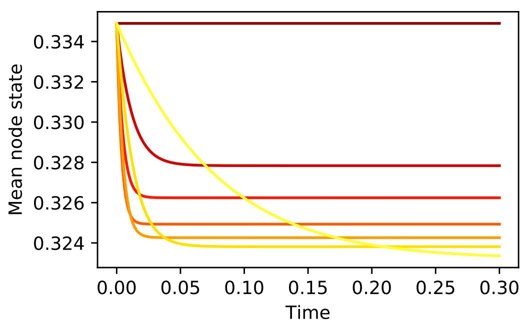

Fig. 2. Mean field evolution of an orbit for a k-edge complete hypergraph of 8 nodes, with $\lambda = 1$, and $k = 2, \ldots, 8$. The initial condition is chosen at random but is fixed between different k. We depict in dark red $k = 2$ and in light yellow $k = 8$ gradually transitioning between this cases.

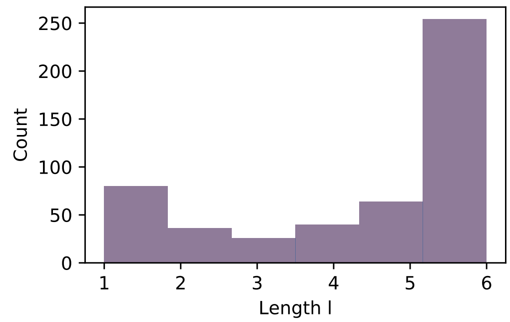

Fig. 3. Histogram of the value l presented in the text for 500 random initial conditions. We observe a much higher concentration in $l = 6$ than the one expected by random chance (a proportion of $2 \cdot 2^{-l} = 2^{-5}$).

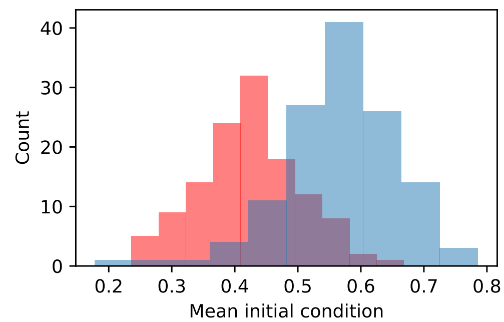

Fig. 4. Histogram of the mean state of the initial condition for the cases in Figure 3 where $l = 6$. In blue the cases where the sequence of $\overline{\mathbf{x}}_k^\infty$ is increasing, and in red the ones where is decreasing.

dynamics are shown in Figure 1. We see that $\overline{\mathbf{x}}$ is not constant but it does not fluctuate much either. We also observe that every component converges to $\overline{\mathbf{x}}$ so we verify our theroretical undersanding that orbits tend to equilibrium points. To see a substantial change in the mean field we take an initial condition with components in $\{0, 1\}$. Taking two null components we get Figure S1 where the change in $\overline{\mathbf{x}}$ is much more significant. However, if we take half components 1 and half 0, we observe that $\overline{\mathbf{x}}$ is constant. This can be explained by the effects of symmetry commented in Eq. (5).

To understand better these dynamics we study separately the ones given by each order of edges, as we expect the general dynamics to be roughly the combination of this ones (which we have confirmed with simulations). For $k \in \{2, \ldots, 8\}$ we consider the complete graph of k-edges with 8 nodes and we study how the mean field changes its behaviour. In Figure 2 it is shown for the random initial conditions from Figure 1 and in Figure S2 it is shown for the initial conditions in Figure S1. In both plots it seems that when $t = +\infty$, the deviation from the initial mean field grows with k. In fact, for binary interactions we see that the mean field is constant, which is what we expected as the system with only two order interactions is reduced to consensus dynamics. We can also see, that when the initial mean field is smaller than 0.5 then the mean field decreases for all orders, and that when the initial mean filed is bigger, it increases.

The observations above hold for most initial conditions but not all of them. To see this given an initial condition, denote by $\overline{\mathbf{x}}_k^\infty$ the convergent point of the mean filed in the complete hypergraph of order k. Now we take 500 random initial conditions and for each one of them find the largest l such that,

$$\overline{\mathbf{x}}_2^\infty \leq \overline{\mathbf{x}}_3^\infty \leq \cdots \leq \overline{\mathbf{x}}_{l+2}^\infty \quad \text{or} \quad \overline{\mathbf{x}}_2^\infty \geq \overline{\mathbf{x}}_3^\infty \geq \cdots \geq \overline{\mathbf{x}}_{l+2}^\infty,$$

and we obtain the histogram in Figure 3. By far the most common occurrence is $l = 6$ which corresponds to the situation depicted in Figure 2 where the values $\overline{\mathbf{x}}_k^\infty$ are monotonic. We study further the cases with $l = 6$, in Figure 4 where we depict the mean state of the initial condition for the cases $\overline{\mathbf{x}}_k^\infty$ increasing, and decreasing separately. We confirm that there is a tendency of having higher mean initial condition when $\overline{\mathbf{x}}_k^\infty$ is increasing and a lower one when it is decreasing.

Directional effects. Although we are considering undirected hypergraphs, the topology given by the higher order edges can allow as to have directional like dynamics. To see this, we consider two complete hypergraphs of 8 nodes, $\mathcal{H}_1$ and $\mathcal{H}_2$, connected by a single edge which has all nodes of $\mathcal{H}_1$ an a single one from $\mathcal{H}_2$. Note that this set up is simpler than the one we would have if we only dealt with 3-way interactions, as then we would need to choose two nodes from the source and one from the target. We initialize all node states in $\mathcal{H}_1$ with 1 and in $\mathcal{H}_2$ with 0. The effect of this edge on the node in $\mathcal{H}_2$ is greatly amplified as all other nodes in the edge have the same state and hence the modulating factor is maximal $s(0)$ (s is decreasing). For nodes in $\mathcal{H}_1$ the modulating constant is $s(1/7)$ and hence much smaller, specially when λ is large. Thus, we will see an unbalanced influence of this edge, which will make the initial state of $\mathcal{H}_1$ to dominate the one in $\mathcal{H}_2$

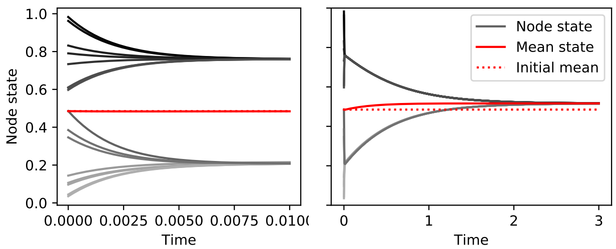

Fig. 5. Coordinates of an orbit in group reinforcing dynamics for two complete hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by a 9-edge with a single node in $\mathcal{H}_2$. We take $\lambda = 1$ and the initial condition is chosen at random in the range $[0.5, 1]$ in $\mathcal{H}_1$ and in the range $[0, 0.5]$ in $\mathcal{H}_2$. The intensity of the grey increases with the value of the initial condition of the coordinate.

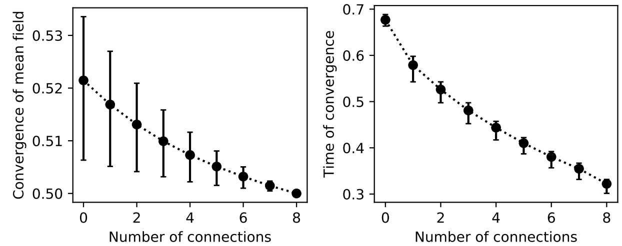

Fig. 6. We do 20 experiments in the following setup and represent the convergent point of the mean field in the left and the time for all coordinates to get within 10^{-5} of the mean in the right. Consider two complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by all 9-edge from $\mathcal{H}_1$ to $\mathcal{H}_2$, with $\lambda = 1$ and initial condition chosen at random as explained in the text. Then, add $k = 0, \ldots, 8$; 9-edges from $\mathcal{H}_2$ to $\mathcal{H}_1$ and consider the group reinforcement dynamics in this hypergraph.

as show in Figure S3. The same principle holds if we pick an initial condition uniformly at random but with range $[0.5, 1]$ for $\mathcal{H}_1$ and $[0, 0.5]$ for $\mathcal{H}_2$ as displayed in Figure 5. The dynamics depicted in it also reveal the modular structure of the hypergraph, first reaching consensus in each module and then reaching global consensus. Note that this is done in a clear separation of time scales exhibiting a slow-fast dynamics.

An alternative model is to take the modules $\mathcal{H}_1$ and $\mathcal{H}_2$ as complete graphs, i.e. with only 2-edges. The main advantage of this module is that as in each module we have consensus dynamics, all changes in $\overline{\mathbf{x}}$ are directly caused by the edge between the modules. As a drawback, the strength of interaction in a module is weaker, and as shown in Figure S4 the node in $\mathcal{H}_2$ directly connected to $\mathcal{H}_1$ converges to $\overline{\mathbf{x}}$ much faster than the rest of nodes in $\mathcal{H}_2$. Hence, the dynamics do not exhibit the modular structure as clearly as in the previous example.

We now want to see how adding more connection between the complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$, changes the convergent point and rate of convergence of $\overline{\mathbf{x}}$. To do so we take 20 initial conditions uniformly at random in $[0, 0.5]$ for $\mathcal{H}_2$ and take one minus this initial conditions for the initial conditions in $\mathcal{H}_1$. In doing this we make sure that the initial mean field is 0.5 and that the situation in each module is equivalent, which makes it easier to appreciate the effects of the topology. For each of this conditions, we compute the convergent point of the mean field and the time it takes for all components to get with a certain tolerance to it (we take tolerance of 10^{-5}).

In Figure S5 we see the case when we add $(N+1)$-edges from $\mathcal{H}_1$ and $\mathcal{H}_2$, i.e. with only one node in $\mathcal{H}_2$, until having all of them. We see that the convergent point of the mean field does not change significantly. However, the time it takes to converge dramatically decreases as the number of edges increases. This is what we expect as more edges accelerate the effect that $\mathcal{H}_1$ has on $\mathcal{H}_2$. In Figure 6 we start with the final configuration of the previous case, and start adding $(N+1)$-edges from $\mathcal{H}_2$ to $\mathcal{H}_1$. In this case, both the distance from the initial mean field and the variability from initial conditions diminish as we add edges. This culminates in the last configuration when we have all $(N+1)$-edges in both directions. In this case, $\overline{\mathbf{x}}$ remains constant in time for any initial condition. This can be explained by the symmetry in the topology and initial conditions, as developed in Eq. (5). The time to converge also decreases as more edges are added.

Finally, we study how the parameter λ effects the dynamics of the system. We take the complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ with

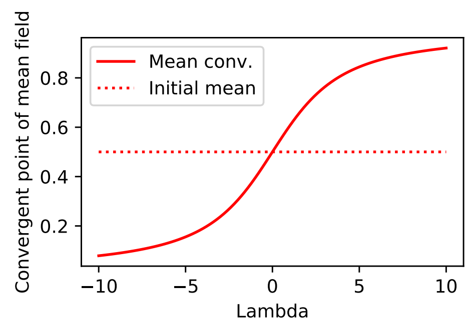

Fig. 7. Convergence point of the mean field, depending on the value λ for two complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by all 9-edge from $\mathcal{H}_1$ to $\mathcal{H}_2$, with initial condition 1 in $\mathcal{H}_1$ and 0 in $\mathcal{H}_2$.

all $(N+1)$-edges from $\mathcal{H}_1$ to $\mathcal{H}_2$. We initialize nodes in $\mathcal{H}_1$ with 1 and nodes in $\mathcal{H}_2$ with 0 to maximize the directional effect. In Figure 7 we can see the convergent point of the mean field for different values of λ. Note that when $\lambda < 0$, then s is an increasing function and hence the directional effects go in the opposite direction. That is, the initial condition in $\mathcal{H}_2$ dominates the one in $\mathcal{H}_1$. Also for $\lambda > 0$, as λ increases, the function s decreases faster, which makes the directional effect stronger, as it is shown in Figure 7.

Conclusions and Discussion

Consensus dynamics is the basic model of network dynamics, which have been key to model various real life scenarious such as epidemics. Its simplicity allows it to be very well understood theoretically. However, it is not flexible enough to allow for more that 2 body interactions, which limits its applicability. In this paper we propose two ways to generalise this dynamics to hypergraph, Hodge Laplacian dynamics and group reinforcement dynamics. These models allow for higher order interactions, and hence can be used in a wider spectrum of applications than the consensus model.

The Hodge Laplacian dynamics are based on the Hodge Laplacian matrix, a generalization of the Laplacian matrix of a graph to simplicial complexes. Its main advantage is that it is a linear system defined by a semi-positive defined matrix, and hence many of the consensus dynamics properties are preserved. This makes it formally, the most natural generalization to consensus dynamics. It also makes it computationally and theoretically easy to understand. For instance, we have been able to show that given an initial condition, an appropriate projection gives the limit point of this orbit. Moreover, the set of limit points can be interpreted as topological invariants of the simplicial complex. However, in this dynamics state variables are given to each simplex. This leads to challenges in analysis and interpretability of the model, as in empirical data states are usually exclusively defined on nodes. One possible solution to this limitation would be to introduce functions which from an initial condition only in nodes generate a state in all simplexes, and others that from a global state collapse it to a node state. One obvious candidate in the first direction would be the mean over nodes.

The group reinforcement model, introduces a non-linear function (suppose now it is decreasing) to ponderate the standard consensus dynamics. Given two nodes i, j in the same edge, this ponderation strengthens the effect of j to i if the state in j is similar to the mean state of the elements of the edge excluding i and weakens it otherwise. This was originally proposed in (11) for 3-way interactions. Here we have expanded their work to deal with a general hypergraph. Moreover, we have been able to show theoretically that this generalization has the same equilibrium points as consensus dynamics and that all orbits converge to one of them. We have also seen numerically how the topology and the initial conditions can influence the convergent point of the mean field. Additionally, we have shown how this topology may cause directional effects, even if the underlining structure is undirected.

The main advantage of this model is that it only deals with nodes states and hence it can be easily implemented to model real life scenarios. However, the need to introduce non-linearity to observe non-reducible multi-body dynamical phenomena makes it difficult to study the system theoretically. For instance, we have observed that for this dynamics the mean field may not be constant, as is the case in the consensus framework. This non-linearity can also be used to model more complex situations, specially if we introduce different modulating functions s for each order. Studying which are reasonable functions to choose would be a good way to expand our work. On this note is important to point out that we have constrained our numerical simulations to the case $s(x) = e^{-\lambda x}$, but it would be interesting to study how other functions may change the dynamics. An interesting candidate would be the Heaviside function, which is 0 for smaller values than a threshold constant, and 1 otherwise, as it is not positive, which makes our theoretical understanding of its dynamics weaker.

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Supporting information (SI)

A. APPEND: Dynamics on simplical complex. Using the maps in Eq. (1) the most natural ODE to define on $\oplus_{k=0}^{N} C_k$ is,

$$\dot{\mathbf{x}}_k = d_k^T \mathbf{x}_{k-1} + d_{k+1}\mathbf{x}_{k+1},$$

We can encode this in a symmetric matrix D, such that we are considering the ODE, $\dot{\mathbf{x}} = D\mathbf{x}$, where $\mathbf{x} = (\mathbf{x}_0, \ldots, \mathbf{x}_N)$. Using the fact that $d_k d_{k+1} = 0$ for all k it is easy to check that the Hodge Laplacian dynamics are given by

$$\dot{\mathbf{x}} = -D^2\mathbf{x}.$$

As D is symmetric it is diagonalisable and hence the spectrum of D^2 is just the squares of the spectrum of D. Despite this, these matrix may define quite different dynamics as D can have both positive and negative eigenvalues.

B. Figures.

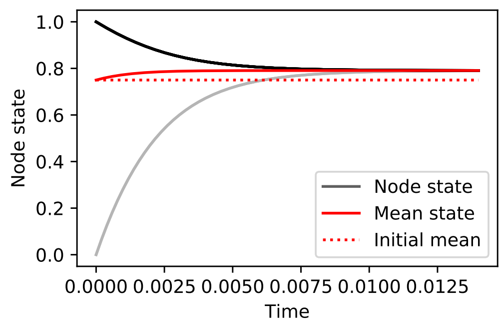

Fig. S1. Coordinates of an orbit in group reinforcing dynamics for a complete hypergraph of 8 nodes, with $\lambda = 1$. The initial condition is $x_1, x_2 = 0$ and the rest or coordinates at 1. The intensity of the grey increases with the value of the initial condition of the coordinate.

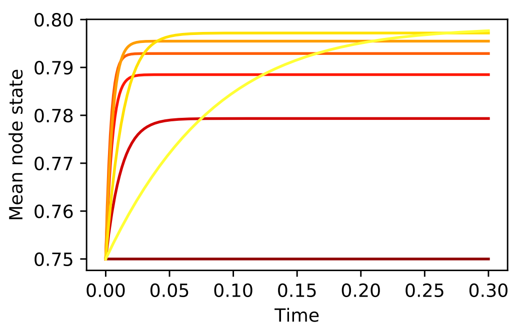

Fig. S2. Mean field evolution of an orbit for a k-edge complete hypergraph of 8 nodes, with $\lambda = 1$, and $k = 2, \ldots, 8$. The initial condition is chosen as $x_1, x_2 = 0$ and the rest of coordinates at 1. We depict in dark red $k = 2$ and in light yellow $k = 8$ gradually transitioning between this cases.

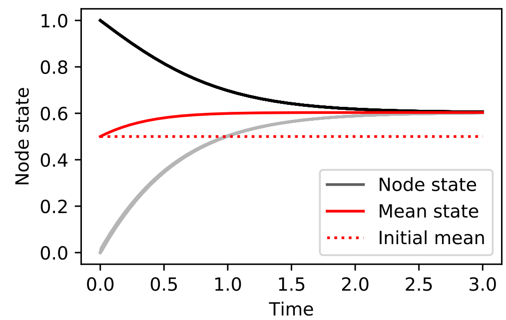

Fig. S3. Coordinates of an orbit in group reinforcing dynamics for two complete hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by a 9-edge with a single node in $\mathcal{H}_2$. We take $\lambda = 1$ and the initial condition is chosen as 1 in $\mathcal{H}_1$ and as 0 in $\mathcal{H}_2$. The intensity of the grey increases with the value of the initial condition of the coordinate.

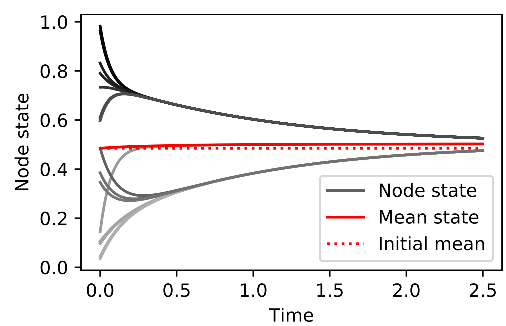

Fig. S4. Coordinates of an orbit in group reinforcing dynamics for two complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by a 9-edge with a single node in $\mathcal{H}_2$. We take $\lambda = 1$ and the initial condition is chosen at random in the range $[0.5, 1]$ in $\mathcal{H}_1$ and in the range $[0, 0.5]$ in $\mathcal{H}_2$. The intensity of the grey increases with the value of the initial condition of that coordinate.

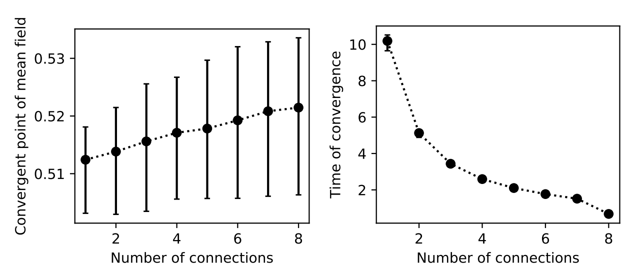

Fig. S5. We do 20 experiments in the following setup and represent the convergent point of the mean field in the left and the time for all coordinates to get within 10^{-5} of the mean in the right. Consider two complete graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ of 8 nodes connected by $k = 1, \ldots, 8$; 9-edge from $\mathcal{H}_1$ to $\mathcal{H}_2$, with $\lambda = 1$ and initial condition chosen at random as explained in the text. Then, consider the group reinforcement dynamics in this hypergraph.

C. Main code. Here we attach the main functions used to do the numerical simulations. The full script of them can be found in the attached `.ipynb` field.

```python
def powerset(s):
    '''Takes a list s and gnerates powerset of s in list format.'''
    x = len(s)
    masks = [1 << i for i in range(x)]
    for i in range(1 << x):
        yield [ss for mask, ss in zip(masks, s) if i & mask]

def s(x): #Function s from the paper, that is used to define ODE
    return np.exp(-1*abs(x))

def index_list(HGraph,nodes):
    '''Given a hypergraph (nodes,HGraph) it returns a list of all the index sets H_i explained in the paper.'''
    return [[[j for j in A if j != i] for A in HGraph if i in A] for i in nodes]

def F(x,t,I_list,s):
    '''Given a point x, a list of the indicator index of a hypergraph (generated by index_list)  and a function s
     , returns  the vector of the flow at x of the ODE defined in the paper. The variable t is not used but is
     needed to be able to use odeint lateron'''
    out=[0 for i in x]
    for i,A_list in enumerate(I_list):
        for A in A_list:
            y=[x[i] for i in A]
            Mean=np.mean(y)
            out[i]+=sum([s(x[j]-Mean)*(x[j]-x[i]) for j in A])
    return out

def scale_color(a):
    '''Given a value a in [0,1] returns the value of the cm.Greys scaleted such that 0 goes to grey and not white
     .'''
    cmap = cm.Greys
    return cmap(a*0.6+0.399)

def ODE_plot(sol_ODE,T=[]):
    '''Given a list of points, sol_ODE, represents the line graphs of each cordinate and the mean. If the
     parameter T is given, it uses it as the x_axes.'''
    if len(T)==0:
        x_axe=list(range(len(sol_ODE)))
        plt.xlabel("Time steps")
    else:
        x_axe=T
        plt.xlabel('Time')
    y_axe=list(map (np.mean,sol_ODE))
    ll=list(zip(*sol_ODE))
    for i in ll:
        plt.plot(x_axe,i,color=scale_color(i[0]))
    grey_line = mlines.Line2D([], [], color=scale_color(0.5),label="Node state")
    red_line,=plt.plot(x_axe,y_axe,color='red',label="Mean state")
    red_line2,=plt.plot(x_axe,[y_axe[0] for i in x_axe],color='red',linestyle=':',label="Initial mean")
    plt.ylabel('Node state')
    plt.legend(handles=[grey_line,red_line,red_line2])
    plt.show()

def ODE_convergence (x0,I_list,s,TOL=1E-10,STEP=1):
    '''Given and initial condition x0, a index list of a hypergraph I_list, and a function s; it computes the
     convergence point of the ODE at infinity and returns the mean of its components and the time speed to reach
     this point with a tolerance TOL and a STEP used to search for the convergence in intervals of this length.
     '''
    x=x0
    T=np.linspace(0,STEP,200)
    sol_ODE=odeint(F,x,T,(I_list,s))
    x=sol_ODE[-1]
    while max(map(abs,x-np.mean(x)))>TOL:
        T=np.linspace(T[-1],T[-1]+STEP,200)
        sol_ODE=odeint(F,x,T,(I_list,s))
        x=sol_ODE[-1]
    mean=np.mean(x)
    ll=[max(map(abs,i-np.mean(i))) for i in sol_ODE]
    ll=[i>TOL for i in ll]
    index=ll.index(False)
    return (mean,T[index])
```